On Criteria for imaginary roots *

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§313 In the preceding chapter we exhibited a method to explore the nature of roots of any equation such that by means of it, if any arbitrary equation is propounded, one can find out, how many real and imaginray roots it has. In most cases, this investigation is certainly most difficult, if the differential equation is of such a nature that its roots cannot be exhibited. But although in these cases the same operation could be applied to the differential equation itself and the nature of its roots could be explored from its differential and hence the roots of the original equation could be assigned approximately, the work would nevertheless be too cumbersome in almost every case. Therefore, in this case it often suffices to know criteria from which, if the their conditions are satisfied, one can conclude that the propounded equation contains imaginary roots, even though, if the conditions are not satisfied, one can vice versa not infer that all roots are real. Even if the knowledge is not complete then, it will be useful very often; therefore, we dedicated the present chapter to the explanation of these criteria.

§314 In the preceding chapter we saw, if any arbitrary equation

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

has only real roots, that its differential

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$$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0$$

will have also only real roots. But at the same time we showed, even though the differential equation only has real roots, hence it nevertheless does not follow that all roots of the propounded equation are real. Nevertheless, if the differential equation has imaginary roots, we will always be able to conclude that the propounded equation must at least have as many imaginary roots, where *at least* is to be stressed; for, it can happen that the equation has more imaginary roots. Therefore, this way from the differential equation one can not conclude more than, if it has imaginary roots, that the propounded equation must also have roots of such a kind, and at least as many.

§315 If the propounded equation is multiplied by any power x^m, while m denotes a positive integer, because this new equation will have only real roots, if all roots of the propounded one were real, then also the roots of its differential, after having divided by x^{m-1}, will all be real. Hence, if this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

has only real roots, also this equation

$$(m+n)x^n - (m+n-1)Ax^{n-1} + (m+n-2)Bx^{n-2} - \text{etc.} = 0$$

will have only real roots. For the same reason, if this equation is multiplied by x^k and differentiated again, the resulting equation

$$(m+n)(k+n)x^n - (m+n+-1)(k+n-1)Ax^{n-1} + (m+n-2)(k+n-2)Bx^{n-2} - \text{etc.} = 0$$

will still have only real roots and one can continue arbitrarily far like this. But if an equation of this kind is detected to have imaginary roots, it will be certain at the same time that the propounded equation will have at least as many imaginary roots.

§316 If the propounded equation, before it is differentiated, is multiplied by no power of x, the decision is to be made for an equation of one degree lower. So, if the propounded equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = 0$$

has only real roots, all roots of its differentials of all orders will also be real. Hence also the roots of all the following equations will be real

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0,$$
$$n(n-1)x^{n-2} - (n-1)(n-2)Ax^{n-3} + (n-2)(n-3)Bx^{n-4} - \text{etc.} = 0,$$
$$n(n-1)(n-2)x^{n-3} - (n-1)(n-2)(n-3)Ax^{n-4} + \text{etc.} = 0,$$
$$n(n-1)(n-2)(n-3)x^{n-4} - (n-1)(n-2)(n-3)(n-4)AX^{n-5} + \text{etc.} = 0$$
$$\text{etc.},$$

which equations are reduced to the following forms

$$x^{n-1} - \frac{n-1}{n}Ax^{n-2} + \frac{(n-1)(n-2)}{n(n-1)}Bx^{n-3} - \frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)}Cx^{n-4} + \text{etc.} = 0,$$
$$x^{n-2} - \frac{n-2}{n}Ax^{n-3} + \frac{(n-2)(n-3)}{n(n-1)}Bx^{n-4} - \frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)}Cx^{n-5} + \text{etc.} = 0,$$
$$x^{n-3} - \frac{n-3}{n}Ax^{n-4} + \frac{(n-3)(n-4)}{n(n-1)}Bx^{n-5} - \frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)}Cx^{n-6} + \text{etc.} = 0,$$
$$x^{n-4} - \frac{n-4}{n}Ax^{n-5} + \frac{(n-4)(n-5)}{n(n-1)}Bx^{n-6} - \frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}Cx^{n-7} + \text{etc.} = 0$$
$$\text{etc.}$$

§317 Therefore, this way the decision can be reduced to an equation of given lower degree than the propounded equation. So, if m was any arbitrary number smaller than n, then, if the propounded equation has only real roots, all roots of this equation of degree m will also be real

$$x^m - \frac{m}{n}Ax^{m-1} + \frac{m(m-1)}{n(n-1)}Bx^{m-2} - \frac{m(m-1)(m-2)}{n(n-1)(n-2)}Cx^{m-3} + \text{etc.} = 0.$$

Hence, if one puts $m = 2$, this equation will result

$$x^2 - \frac{2}{n}Ax + \frac{2 \cdot 1}{n(n-1)}B = 0,$$

whose roots have to be real, if the propounded equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = 0$$

has only real roots. But because this quadratic equation can only have real roots, if $\frac{AA}{nn} > \frac{2 \cdot 1}{n(n-1)}B$, it follows that all roots of the propounded equation can only be real, if $AA > \frac{2n}{2n-1}B$. Therefore, if it was $AA < \frac{2n}{2n-1}B$, this will be a certain indication that at least two roots of the propounded equation will be imaginary.

§318 Hence we derived a necessary condition, which the coefficients of the three first terms have to satisfy, if all roots of the propounded equation were real. And this is a criterion of such kind we mentioned at the beginning: Even though in the case $AA > \frac{2n}{n-1}B$ nothing follows for the realness of the roots, if $AA < \frac{2n}{n-1}B$, it will nevertheless be a certain indication for the existence of at least two imaginary roots. So for all roots be real, by successively substituting the numbers 2, 3, 4, 5 etc. for n, it has to be as follows:

$$\begin{array}{ll} x^2 - Ax + B = 0 & A^2 > 4B \\ x^3 - Ax^2 + Bx - C = 0 & A^2 > \frac{6}{2}B \\ x^4 - Ax^3 + Bx^2 - Cx + D = 0 & A^2 > \frac{8}{3}B \\ x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0 & A^2 > \frac{10}{4}B. \end{array}$$

Hence, if the second term is missing and the coefficient B of the third is positive that the equation is of this kind

$$x^n + Bx^{n-1} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

not all roots can be real, but at least two will be imaginary.

§319 Criteria of this kind can indeed be found for the coefficients of the following terms, if we consider that this equation

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

has as many real and imaginary roots as the propounded equation. For, this equation results from the given one, if one puts $x = \frac{1}{y}$ such that from the roots of this equation one at the same time knows the roots of the latter. Hence, if the propounded equation has only real roots, also all roots of the differential equation of the reciprocal equation, i.e. of this one

$$-A + 2By - 3Cy^2 + 4Dy^3 - \text{etc.} = 0,$$

will be real. Substitute x for $\frac{1}{y}$ in this equation again and this equation will result

$$Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} - 4Dx^{n-4} + \text{etc.} = 0,$$

whose roots will therefore be real, if the roots of the propounded equation were real. Hence it is now plain, if it was $n = 3$, that necessarily $BB > 3AC$.

§320 But now differentiate this equation again and these equations will result

$$Ax^{n-2} - \frac{2(n-2)}{n-1}Bx^{n-3} + \frac{3(n-2)(n-3)}{(n-1)(n-2)}Cx^{n-4} - \text{etc.} = 0$$

$$Ax^{n-3} - \frac{2(n-3)}{n-1}Bx^{n-4} + \frac{3(n-3)(n-4)}{(n-1)(n-2)}Cx^{n-5} - \text{etc.} = 0$$

$$Ax^{n-4} - \frac{2(n-4)}{n-1}Bx^{n-5} + \frac{3(n-4)(n-5)}{(n-1)(n-2)}Cx^{n-6} - \text{etc.} = 0$$

etc.

Therefore, in general, if the number m is smaller than n, it will be

$$Ax^m - \frac{2m}{n-1}Bx^{m-1} + \frac{3m(m-1)}{(n-1)(n-2)}Cx^{m-2} - \text{etc.} = 0.$$

If one now puts $m = 2$, one will have this equation

$$Ax^2 - \frac{4}{n-1}Bx + \frac{6}{(n-1)(n-2)}C = 0;$$

for its roots to be real it is necessary that $\frac{4BB}{(n-1)^2} > \frac{6AC}{(n-1)(n-2)}$. Hence, if the propounded equation has only real roots, it will be

$$BB > \frac{3(n-1)}{2(n-2)}AC.$$

And if it was $BB < \frac{3(n-1)}{2(n-2)}AC$, this is a certain sign that the propounded equation has at least two imaginary roots. Therefore, if $n = 3$, the criterion will be $BB > 3AC$; but if $n = 4$, it will be $BB > \frac{3 \cdot 3}{2 \cdot 2}AC$; if $n = 5$, it will be $BB > \frac{3 \cdot 4}{2 \cdot 3}AC$ and so forth.

§321 To transfer these criteria to the following coefficients, let us consider the differential equation expressed in y again

$$-A + 2By - 3Cy^2 + 4Dy^3 - 5Ey^4 + \text{etc.} = 0$$

and let us differentiate it once more that we have

$$2B - 6Cy + 12Dy^2 - 20Ey^3 + \text{etc.} = 0,$$

which, having substituted $\frac{1}{x}$ for y again, will give

$$Bx^{n-2} - 3Cx^{n-3} + 6Dx^{n-4} - 10Ex^{n-5} + \text{etc.} = 0,$$

from whose further differentiation this equation follows

$$Bx^{n-3} - \frac{3(n-3)}{n-2}Cx^{n-4} + \frac{6(n-3)(n-4)}{(n-2)(n-3)}Dx^{n-5} - \text{etc.} = 0$$

and in general

$$Bx^m - \frac{3m}{n-2}Cx^{m-1} + \frac{6m(m-1)}{(n-2)(n-3)}Dx^{m-3} - \text{etc.} = 0.$$

Therefore, if we put $m = 2$, this quadratic equation will result

$$Bx^2 - \frac{2 \cdot 3}{n-2}Cx + \frac{6 \cdot 2}{(n-2)(n-3)}D = 0,$$

whose roots will be real, if it was $\frac{)CC}{(n-2)^2} > \frac{6 \cdot 2BD}{(n-2)(n-3)}$ or

$$CC > \frac{4(n-2)}{3(n-3)}BD.$$

Hence, if the propounded equation has only real roots, it will be $CC > \frac{4(n-2)}{3(n-3)}BD$, and if this condition is not satisfied, the equation will certainly have at least two imaginary roots.

§322 If we differentiate the above equation $2B - 6Cy + 12Dy^2 - \text{etc.} = 0$ once again, this equation will result

$$-6C + 24Dy - 60Ey^2 + \text{etc.} = 0$$

or

$$C - 4Dy + 10Ey^2 - 20Fy^3 + \text{etc.} = 0,$$

which having substituted x for $\frac{1}{y}$ again will go over into this one

$$Cx^{n-3} - 4Dx^{n-4} + 10Ex^{n-5} - 20Fx^{n-6} + \text{etc.} = 0,$$

from whose further differentiation these equations follow

$$Cx^{n-4} - \frac{4(n-4)}{n-3}Dx^{n-5} + \frac{10(n-4)(n-5)}{(n-3)(n-4)}Ex^{n-6} - \text{etc.} = 0,$$

$$Cx^{n-5} - \frac{4(n-5)}{n-3}Dx^{n-6} + \frac{10(n-5)(n-6)}{(n-3)(n-4)}Ex^{n-7} - \text{etc.} = 0$$

and in general

$$Cx^m - \frac{4m}{n-3}Dx^{m-1} + \frac{10m(m-1)}{(n-3)(n-4)}Ex^{m-2} - \text{etc.} = 0.$$

Let us put $m = 2$ and it will be

$$Cx^2 - \frac{2 \cdot 4}{n-3}Dx + \frac{2 \cdot 10}{(n-3)(n-4)}E = 0,$$

from which, if its roots are real, it follows that

$$\frac{4 \cdot 4}{(n-3)^2} > \frac{2 \cdot 10}{(n-3)(n-4)}CE \quad \text{or} \quad DD > \frac{5(n-3)}{4(n-4)}CE.$$

§323 From these results one already clearly sees the relation for all coefficients. Therefore, in general, if this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \text{etc.} = 0$$

has only real roots, it will be

$$\begin{aligned} AA &> \frac{2n}{1(n-1)}B \\ BB &> \frac{3(n-1)}{2(n-2)}AC \\ CC &> \frac{4(n-2)}{3(n-3)}BD \\ DD &> \frac{5(n-3)}{4(n-4)}CE \\ EE &> \frac{6(n-4)}{5(n-5)}DF \\ &\text{etc.} \end{aligned}$$

If one of these conditions is not satisfied, the equation will have at least two imaginary roots. And if these criteria do not depend on each other, it is easily seen that there are as many pairs of imaginary roots as the total amount of non-satisfied conditions. But even if these conditions all hold in one single equation, it hence nevertheless does not follow that no imaginary roots are given; it can even happen, because there is no reason against it, that all roots are imaginary. Therefore, one has to be careful that not more is attributed to these criteria than it can actually attributed to them considering the principles whence they were deduced.

§324 But it is easily seen that not each condition, which is not satisfied, can indicate imaginary roots; for, in an equation of n dimensions, since one has $n+1$ terms and from each, except for the first and the last, a criterion can be derived, one will in total have $n-1$ conditions; and nevertheless, if they are not satisfied, the equation can not have $2n-2$ imaginary roots, since it in total has only n roots. But one condition alone always reveals two imaginary roots, and since it can happen that two conditions of this kind do not show more roots, one has to consider, whether these two conditions indicate the same imaginary root or not; in the first case the number of imaginary roots will not increase, in the second on the other hand, since the conditions involve completely different letters, each one of them will show two imaginary roots. So, even though it was

$$AA < \frac{2n}{1(n-1)}B \quad \text{and} \quad BB < \frac{3(n-1)}{2(n-2)}AC,$$

hence nevertheless not necessarily four imaginary roots are indicated, but both of them might indicate the same two roots. On the other hand, if it was

$$AA < \frac{2n}{1(n-1)B} \quad \text{and} \quad CC < \frac{4(n-2)}{3(n-3)}BD$$

while $BB > \frac{3(n-1)}{2(n-2)}AC$, four imaginary roots will be indicated.

§325 Therefore, from several criteria involving consecutive letters for imaginary roots it does not follow more than from one; but if they proceed in an interrupted order that between each two at least one other criterion was skipped, then from each one of them one can conclude two imaginary roots. This consideration yields the following rule. Except for the first and the last term write the coefficients found from the criteria before over the respective terms of the propounded equation this way

$$\begin{array}{cccccccccc}
 & & \frac{2n}{1(n-1)} & & \frac{3(n-1)}{2(n-2)} & & \frac{4(n-2)}{3(n-3)} & & \frac{5(n-3)}{4(n-4)} & \text{etc.} \\
x^n & - & Ax^{n-1} & + & Bx^{n-2} & - & Cx^{n-3} & + & Dx^{n-4} & - \ \text{etc.} = 0 \\
+ & & \cdots & & \cdots & & \cdots & & \cdots & \text{etc.}
\end{array}$$

Then examine the square of each coefficient, whether it is larger or smaller than the fraction written above it multiplied by the product of the corresponding coefficients; in the first case attribute the sign $+$ to the term, in the second the sign $-$; but always attribute term the sign $+$ to the first and the last. Having done this, the equation will have at least as many imaginary roots as there are variations of the signs.

§326 This is the rule found by Newton to explore the imaginary roots of each equation; but one has to pay attention, as we already mentioned, since it can happen that the equation has more imaginary roots than one detects by means of this method. Hence others tried to find other similar rules which would yield the number of imaginary roots more exactly, such that the true number of roots would exceed the number which the rule gives less often. In this regard, especially the rule of Campbell added to Newtons *Universal Arithmetic* stands out, which will therefore be conveniently explained here, even if it is not perfect. It is based on this lemma: If α, β, γ, δ, ε etc. were some quantities and their total amount is m, put the sum of these quantities

$$\alpha + \beta + \gamma + \delta + \text{etc.} = S,$$

the sum of the squares

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.} = V,$$

and it will be $V > 0$. But because the product of each two is

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \text{etc.} = \frac{SS - V}{2},$$

it will be $(m-1)V > SS - V$ or $mV > SS$. For, if the squares of the differences of two quantities are taken, their sum will be

$$\begin{aligned}
&= (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\alpha-\delta)^2 + (\beta-\gamma)^2 + (\beta-\delta)^2 + \text{etc.} \\
&= (m-1)(\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}) - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\text{etc.}) \\
&= (m-1)V - 2\frac{SS-V}{2} = mV - SS.
\end{aligned}$$

Therefore, since the sum of real squares is always positive, it will be

$$mV - SS > 0 \quad \text{and hence} \quad mV > SS.$$

§327 Having stated this lemma in advance, if one has this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - \text{etc.} = 0$$

and all its n roots were real, which we want to be a, b, c, d, e etc., it will be, as it is known from the nature of equations,

	numbers of terms
$A = a + b + c + d + \text{etc.}$	n
$B = ab + ac + ad + bc + bd + \text{etc.}$	$\frac{n(n-1)}{1 \cdot 2}$
$C = abc + abd + abe + acd + bcd + \text{etc.}$	$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$
$B = abcd + abce + abde + \text{etc.}$	$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$

etc.

Now take the squares of each term and put

$$\begin{aligned}
P &= a^2 + b^2 + c^2 + d^2 + \text{etc.},\\
Q &= a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + \text{etc.},\\
R &= a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + \text{etc.},\\
S &= a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + \text{etc.}\\
&\text{etc.;}
\end{aligned}$$

but from the nature of combinatorics it will be

$$\begin{aligned}
P &= A^2 - 2B,\\
Q &= B^2 - 2AC + 2D,\\
R &= C^2 - 2BD + 2AE - 2F,\\
S &= D^2 - 2CE + 2BF - 2AG + 2H\\
&\text{etc.}
\end{aligned}$$

§328 By means of the lemma stated in advance we will therefore have

$$nP > AA,$$
$$\frac{n(n-1)}{1 \cdot 2} Q > BB,$$

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}R > CC,$$
$$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}S > DD$$
etc.

Therefore, if the values found before are substituted for the values P, Q, R etc., we will obtain the following properties of the real roots

$$nAA - 2nB > AA \quad \text{or} \quad AA > \frac{2n}{n-1}B,$$
$$\frac{n(n-1)}{1\cdot 2}BB - \frac{2n(n-1)}{1\cdot 2}AC + \frac{2n(n-1)}{1\cdot 2}D > BB$$

or

$$BB > \frac{\frac{2n(n-1)}{1\cdot 2}}{\frac{n(n-1)}{1\cdot 2} - 1}(AC - D)$$

and in like manner the following equations yield

$$CC > \frac{\frac{2n(n-1)(n-2)}{1\cdot 2\cdot 3}}{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - 1}(BD - AE + F),$$

$$DD > \frac{\frac{2n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}}{\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} - 1}(CE - BF + AG - H).$$

Therefore, the square of each coefficient is not only compared to the product of the closest terms, but also to the rectangles of two equally distant ones, nevertheless in such a way that the signs of these rectangles alternate.

§329 Therefore, except for the first and the last one has to write the fractions, whose numerators are the binomial coefficients of the same power multiplied by two and whose denominators are the same binomial coefficients decreased by 1, above the respective terms of the equation. So, by considering quadratic, cubic, fourth order equations etc., if their roots all were real, it will be

$$x^2 - \overset{\frac{4}{1}}{A}x + B = 0; \quad A^2 > 4B;$$

For the cubic equation

$$x^3 - \overset{\frac{6}{2}}{A}x^2 + \overset{\frac{6}{2}}{B}x - C = 0$$

it will be

$$A^2 > 3B \quad \text{and} \quad B^2 > 3AC.$$

For the forth order equation

$$x^4 - \overset{\frac{8}{3}}{A}x^3 + \overset{\frac{12}{5}}{B}x^2 - \overset{\frac{8}{3}}{C}x + \ = 0$$

it will be

$$A^2 > \frac{8}{3}B, \quad B^2 > \frac{12}{5}(AC - D), \quad C^2 > \frac{8}{3}BD.$$

For the equation of the fifth order

$$x^5 - \overset{\frac{10}{4}}{A}x^4 + \overset{\frac{20}{9}}{B}x^3 - \overset{\frac{20}{9}}{C}x^2 + \overset{\frac{10}{4}}{D}x - E = 0$$

it will be

$$AA > \frac{10}{4}B, \quad B^2 > \frac{20}{9}(AC - D), \quad C^2 > \frac{20}{9}(BD - AE) \quad \text{and} \quad D^2 > \frac{10}{4}CE.$$

For the equation of the sixth order

$$x^6 - \overset{\frac{12}{5}}{A}x^5 + \overset{\frac{30}{14}}{B}x^4 - \overset{\frac{40}{19}}{C}x^3 + \overset{\frac{30}{14}}{D}x^2 - \overset{\frac{12}{5}}{E}x + F = 0$$

it will be

$$A^2 > \frac{12}{5}B, \quad B^2 > \frac{30}{14}(AC - D), \quad C^2 > \frac{40}{19}(BD - AE + F),$$
$$D^2 > \frac{30}{14}(CE - BF), \quad E^2 > \frac{12}{5}DF.$$

etc.

§330 Therefore, if a certain condition is not satisfied, it will be an indication that at least two imaginary roots are contained in the propounded equation. But because, if each condition is not satisfied, the equation can not have twice as many imaginary roots, one has to argue in these cases as before in the case of Newton's rule. If the square of a certain term was larger than the fraction written above it multiplied by the product of the closest and equally distant terms, then attribute the sign $+$ to this term, otherwise the sign $-$; but always attribute the sign $+$ to the first and the last term. Having done this, check the progression of the signs, and as often as a variation occurs, an imaginary root will be indicated. Therefore, if this rule indicates more imaginary roots than Newton's rule, it will be closer to the truth. Nevertheless, it can happen that the equation has more imaginary roots than each of the rules indicates.

§331 Therefore, we would make a mistake, if we wanted to use these criteria as perfect indication for real and imaginary roots, since it can happen that the equation has more imaginary roots than these criteria indicate; and the error could be the greater, the higher the degree of the propounded equation was. For, in the case of quadratic equations these criteria are true in such a way that, if they do not indicate any imaginary roots, the equation will also have none. But the cubic equation can have two imaginary roots, even though both rules (they coincide in this case) do not exhibit them. Therefore, to someone wanting to investigate these cases, let this general cubic equation be propounded

$$x^3 - \overset{3}{Ax^2} + \overset{3}{Bx} - C = 0;$$

if in this it was $AA > 3B$ and $BB > 3AC$, none of both rules indicates imaginary roots. But above (§ 306) we saw that for this equation to have imaginary roots, at first it is required that $B < \frac{1}{3}AA$, which condition also both rules require. Therefore, let $B = \frac{1}{3}AA - \frac{1}{3}FF$ and it is necessary that C is contained within these limits

$$\frac{1}{27}A^3 - \frac{1}{9}AFF - \frac{2}{27}f^2 \quad \text{and} \quad \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

But both rules only demand that $C < \frac{BB}{3A}$, i.e.

$$C < \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A}.$$

This condition can be satisfied, even though C is not contained within the mentioned limits.

§332 For, let

$$C = \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg$$

and the rules will indicate no imaginary roots. There will nevertheless be imaginary roots, if it was either

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$$

or

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg > \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

Therefore, if it was either

$$gg > \frac{(ff+Af)^2}{27A} \quad \text{or} \quad gg < \frac{(Af-ff)^2}{27A},$$

the cubic equation will have two imaginary roots, even though none of both rules indicates them. But here we assumed that A is a positive quantity; for, if it was negative, by putting $x = -y$ the equation would be transformed into a form in which A would be positive. Hence one can form infinitely many cubic equations, which have two imaginary roots, even though they are not indicated by the rule. For, let $gf = \frac{(ff+Af)^2}{27A} + hh$; it will be

$$C = \frac{(ff-AA)^2}{27A} - gg = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - hh \quad \text{and} \quad B = \frac{1}{3}AA - \frac{1}{3}ff.$$

Or let it be $gg = \frac{(Af-ff)^2}{27A} - hh$ with $hh < \frac{(Af-ff)^2}{27A}$; it will be

$$C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 + hh \quad \text{and} \quad B = \frac{1}{3}AA - \frac{1}{3}ff.$$

In both cases an equation having two imaginary roots, which none of both rules indicate, will result. For the sake of an example let us put $A=4$, $f=1$; it will be $B=5$ and, because of $gg=\frac{25}{108}+hh$, it will be

$$C=\frac{225}{108}-\frac{25}{108}-hh=\frac{50}{27}-hh.$$

Hence, if $C<\frac{50}{27}$, the equation $x^3-4x^2+5x-C=0$ will always have two imaginary roots. But having taken $gg=\frac{1}{12}-hh$, it must be $hh<\frac{1}{12}$ and it will be

$$C=\frac{25}{12}-\frac{1}{12}+hh=2+hh.$$

Let $hh=\frac{1}{16}$ and the equation $x^3-4x^2+5x-\frac{33}{16}=0$ will have two imaginary roots, even though none is revealed by the rules.

§333 It is even possible to form general equations of such a kind, in which none of both rules exhibits imaginary roots, even though in most cases two or more are contained in the equation. This happens, if two equal signs alternate, as in

$$x^n-Ax^{n-1}-Bx^{n-2}+Cx^{n-3}+Dx^{n-4}-Ex^{n-5}-Fx^{n-6}+\text{etc.}=0$$

or

$$x^n+Ax^{n-1}-Bx^{n-2}-Cx^{n-3}+Dx^{n-4}+Ex^{n-5}-Fx^{n-6}-\text{etc.}=0;$$

here, both rules do not reveal an imaginary root. But that they most often can contain roots of this kind, is also clear from the cubic equation $x^3-Ax^2-Bx+C=0$, which for $ff=AA+3B$ always has two imaginary roots, if it was either

$$-C<\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3 \quad \text{or} \quad -C>\frac{1}{27}A^3-\frac{1}{9}Aff+\frac{2}{27}f^3.$$

Nevertheless, also these cases can be found from the rules, if the equation is transformed into another form by means of a substitution. Put $x=y+k$ and it will be

$$\left.\begin{array}{llll} y^3 + 3ky^2 - 3kky + & k^3 & & \\ \quad - Ayy - & 2Aky & - Akk & \\ \quad - & By & - Bk & \\ & & + C & \end{array}\right\} = 0,$$

which examined according to the rule will first immediately give

$$(3k - A)^2 > 3(3kk - 2Ak - B);$$

but for it to be

$$(3kk - 2Ak - B)^2 > 3(3k - A)(k^3 - Akk - Bk + C),$$

which is the other criterion, it is necessary that

$$BB + 3AC + (AB - 9C)k + (AA + 3B)kk > 0,$$

whatever value is attributed to k. Therefore, choose k in such a way that this expression has minimum value, what will happen for $k = \frac{9C-AB}{2(AA+3B)}$, and if this expression was still > 0, it will be probable that the propounded equation has no imaginary roots. But it will be

$$BB + 3AC - \frac{(AB - 9C)^2}{2(AA + 3B)} + \frac{(AB - 9C)^2}{4(AA + 3B)} > 0$$

or

$$BB + 3AC > \frac{(AB - 9C)^2}{4(AA + 3B)}.$$

Therefore, since $B = \frac{1}{3}ff - \frac{1}{3}AA$, it will be

$$4ff\left(\frac{1}{9}f^4 - \frac{2}{9}AAff + \frac{1}{9}A^4 + 3AC\right) > \left(\frac{1}{3}Aff - \frac{1}{3}A^3 - 9C\right)^2$$

or

$$4f^6 - (A^2f^4 + 4A^4ff + 108ACff > A^2f^4 - 2A^4f^2 - 54ACff + A^6 + 54A^3C + 729CC$$

or

$$4f^6 > 9A^2f^4 - 6A^4ff - 162ACff + A^6 + 54A^3C + 729CC,$$

whence, having factored the equation, it will have to be

$$(2f^3 + A^3 - 3Af^2 + 27C)(2f^3 - A^3 + 54A^3C + 27C) > 0.$$

And hence the rules will show imaginary roots, if it was either

$$C > -\frac{1}{27}A^3 + \frac{1}{9}Af^2 - \frac{2}{27}f^3 \quad \text{and} \quad C > -\frac{1}{27}A^3 + \frac{1}{9}Af^2 + \frac{2}{27}f^3$$

$$C < -\frac{1}{27}A^3 + \frac{1}{9}Af^2 - \frac{2}{27}f^3 \quad \text{and} \quad C < -\frac{1}{27}A^3 + \frac{1}{9}Af^2 + \frac{2}{27}f^3.$$

These are the same conditions which we found above [§ 306]. Therefore, it is plain that by means of an appropriate transformation the rules given in this chapter can be stated in such a way that they are always true, even though they are converted.

§334 From these principles also Harriot's rule can be demonstrated, according to which any arbitrary equation is predicted to have as many positive roots as there are variations of the signs, but as many negative as there are successions of the same sign; but this rule only holds for real roots. Therefore, let us put that the equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

has only real and positive roots and its differential

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - \text{etc.} = 0$$

will have not only also only real and positive roots, but the roots of this one will also constitute the limits of the roots of the propounded equation. Furthermore, having put $x = \frac{1}{y}$, this equation

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

will have only real positive roots, but they are the reciprocals of the others, such that the roots, which are the maxima in that equation, are the minima in this one. Having constituted all this, if that propounded equation is continuously differentiated until one gets to an equation of first order, which will be $x - \frac{1}{n}A = 0$ (§ 317), the root of this one will still be positive and hence the coefficient of the second term will have the sign $-$, as we assumed. But if this coefficient would have the sign $+$, it would follow that the propounded equation has not only positive real roots, but at least one will be negative, of course the one, which corresponds to the mentioned limits.

§335 If the propounded equation is converted into its reciprocal and is differentiated, then x is substituted again and the differentiations are repeated until one gets to a simple equation, which from § 320 will be of this kind $Ax - \frac{2}{n-1}B = 0$, its roots must therefore also be positive, if the propounded equation has only real and positive roots, and hence the second and the third term will have different signs. Therefore, if these two terms have the same sign, at least one negative root will be indicated corresponding to the limit assigned in this equation, which will be different from the limit indicated by the equation, since here they were once converted into its reciprocals; hence one concludes, if the three initial terms of the equation have equal signs, that two negative roots will be indicated.

§336 In like manner, if the conversions and differentiations are done according to § 320 and are continued until one gets to the simple equation $Bx - \frac{3}{n-2}C = 0$, also the roots of this equation must be positive, if all roots of the propounded equation were positive, of course; hence, if the third and the fourth term have equal signs, one negative root will be indicated. And so forth, if any two contiguous terms have the same sign, one negative root will be indicated; and hence, no matter how many successions of the same sign occur, the propounded equation will have at least as many negative roots, since each of these criteria refer to different limits. But if the propounded equation is put to have only negative roots, then, because the roots of all differential equations deduced from it must also be negative, all terms need to have the same sign. Hence, if two contiguous terms have different signs, from them at least one positive root will be concluded. And in like manner, no matter how many variations of the signs of two terms occur in the propounded equation, at least as many positive roots are concluded to be contained in the equation. Therefore, since each equation has as many roots as there are combinations of two contiguous signs, and not more, it follows that each equation, whose roots are all real, has as many positive roots as there were variations of contiguous signs, but has as many negative roots as there were iterations of the same sign.